Linear Algebraic Equations

Read
- Matrix notations
- Matrix algebra

row vector $\lfloor C \rfloor$
column vector $\{C\}$

Identity matrix

Banded matrix

tridiagonal matrix →

trace of a Matrix

$$tr[A] = \sum_{i=1}^{n} a_{ii}$$

sum of elements on its Principal diagonal.

augmentation

$$[A] = \begin{bmatrix} 5 & 6 & 7 \\ 1 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$[A] = \left[\begin{array}{ccc|cc} 5 & 6 & 7 & 3 & 4 \\ 1 & 2 & 5 & 1 & 2 \\ 1 & 1 & 3 & 1 & 3 \end{array}\right]$$

Linear algebraic equations in matrix from

$$[A]\{x\} = \{B\}$$

known | find | known

$$[A]^{-1}[A]\{x\} = [A]^{-1}\{B\}$$

$$\{X\} = [A]^{-1}\{B\}$$

Augment A with B

Elimination of unknowns

$$\begin{aligned} 5x + 6y &= 3 \quad 2 \\ 2x + 3y &= -2 \quad 5 \end{aligned}$$

Naive Gauss Elimination

{ Elimination
{ Back substitution

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n = b_1 \quad -1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = b_2 \quad -2$$

$$\vdots$$

$$a_{n1}x_1 + \ldots\ldots = b_n \quad -n$$

multiply $\frac{a_{21}}{a_{11}}$ to 1

Subtract it from 2

$$\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right) x_2 + \cdots\cdots + \left(a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}\right) x_n$$

$$= b_2 - \frac{a_{21}}{a_{11}} b_1$$

$$a'_{22} x_2 + \cdots\cdots + a'_{2n} x_n = b'_2$$

Eq 1 is the pivot equat^n

a_{11} — pivot element

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a''_{33} & b''_3 \end{array}\right]$$

$$x_3 = \frac{b''_3}{a''_{33}}$$

$$x_2 = \left(b'_2 - a'_{23} x_3\right) / a'_{22}$$

$$x_1 = \left(b_1 - a_{12} x_2 - a_{13} x_3\right) / a_{11}$$

$$x_n = \frac{b_n^{n-1}}{a_{nn}^{n-1}}$$

$$x_i = \frac{b_i^{i-1} - \sum_{j=i+1}^{n} a_{ij}^{i-1} x_j}{a_{ii}^{i-1}} \quad \text{for } i = n-1, n-2 \cdots 1$$

Operation counting

(flops)

floating point operations

$$\sum_{i=1}^{m} 1 = 1+1+1 \cdots = m$$

$$\sum_{i=K}^{m} 1 = m-K+1$$

$$\sum_{i=1}^{m} i = 1+2+3+ \cdots = \frac{m(m+1)}{2} = \frac{m^2}{2} + O(m)$$

$\frac{2n^3}{3} + O(n^2)$ $\quad n^2 + O(n)$

n	Elimination	Back substitution	Total flops	$\frac{2n^3}{3}$
10	705	100	805	667
100	671550	10,000	681550	666667
1000	6.67×10^8	1×10^6	6.68×10^8	6.67×10^8

1- as n grows no. of computation grows.

2- Elemination requires more ~~effor~~ computations.

Pitfalls

1. Division by zero
2. round off errors
3. ill conditioned system / a small change in coeffs. result in large changes in the solution.

Ex:

$x_1 + 2x_2 = 10$ $1.1x_1 + 2x_2 = 10.4$	4, 3
$x_1 + 2x_2 = 10$ $1.05x_1 + 2x_2 = 10.4$	8, 1

Example 9.7.

Scaling

Singular System

Determinant of a singular system is zero.

Determine the largest available coeff. in the col. below the pivot element. Switch over.

Partial Pivoting.

Forward Elimination
Back Substitution
Pivoting

Gauss – Jordan

1. The unknown is eliminated from all other equations
2. All rows are normalized, (divide by pivot element)
   Elimination results in an identity matrix.
3. No need to back substitute

Steps
1- normalize first row (divide by pivot element)
2- eliminate x_1 from all except 1^{st} row
3. normalize 2^{nd} row
4. eliminate x_2 from all except 2^{nd} row
5. normalize 3^{rd} row
6. eliminate x_3 from 1 & 2.

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

① $$\begin{bmatrix} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

② $$\begin{bmatrix} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

③ $$\begin{bmatrix} 1 & -0.033 & -0.0667 & 2.6167 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{bmatrix}$$

④ $$\begin{bmatrix} 1 & -0.033 & -0.0667 & 2.6167 \\ 0 & 1 & -0.04188 & -2.793 \\ 0 & -0.19 & 10.02 & 70.61 \end{bmatrix}$$

⑤ $$\begin{bmatrix} 1 & 0 & -0.068 & 2.52 \\ 0 & 1 & -0.041 & -2.79 \\ 0 & 0 & \underline{10.012} & 70.08 \end{bmatrix}$$

⑥ $$\begin{bmatrix} 1 & 0 & -0.068 & 2.52 \\ 0 & 1 & -0.041 & -2.79 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

⑦ $$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2.5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Flops
$$n^3 + n^2 - n \rightarrow n^3 + O(n^2)$$